

ON CAKE DIVIDING

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ABSTRACT. Considered Steinhaus geometric game on cake dividing.

Hugo Steinhaus in his popular book [S] considered the following game (problem 51). Pavel and Havel are dividing a cake as follows. Pavel selects a point P and Havel draws a line l through the point P and gets his piece of the cake. What form of the cake is most advantageous for Havel and which part of the cake he obtains in this case?

Now we consider the formal interpretation of the game.

Let μ be the Lebesgue measure on the space \mathbb{R}^n . Let \mathcal{B}^n be a family of compacts of the space \mathbb{R}^n with non zero measure. For every point $a \in \mathbb{R}^n$ put $l_a = \{x \in \mathbb{R}^n : (x, a) \geq 0\}$. Put

$$b_n = \inf_{A \in \mathcal{B}^n} \sup_{x \in A} \inf_{a \in \mathbb{R}^n} \frac{\mu(A \cap (x + l_a))}{\mu(A)}.$$

Steinhaus proved that $1/4 \leq b_2 \leq 1/3$.

Theorem 1. *For every n holds $b_n = 1/(n+1)$.*

Proof. Let $S \subset \mathbb{R}^n$ be the regular simplex with vertices a_0, \dots, a_n . Every point $a \in S$ has an unique representation $a = \sum \lambda_i a_i$ such that $\lambda_i \geq 0$ for every i and $\sum \lambda_i = 1$. Put $S_i = a_i - S$ for every i . If $i \neq j$ then $S_i \cap S_j = \{0\}$. Indeed suppose that there exist points $s_i, s_j \in S$ such that $x = a_i - s_i = a_j - s_j$. Let $s_j = \sum \lambda_{ik} a_k$ and $s_j = \sum \lambda_{jk} a_k$ be the representations of the points s_i and s_j . The unicity of the representation of the point $(a_i + s_j)/2 = (a_j + s_i)/2$ gives that $1 + \lambda_{ji} = \lambda_{ii}$. Thus $\lambda_{ii} = 1$, $s_i = a_i$ and $x = 0$.

Put $A = \bigcup S_i$. Then $\mu(A) = (n+1)\mu(S)$. Let $a \in S$, $a = \sum \lambda_i a_i$ be the representation of the point a and $\lambda_j = \max \lambda_i$. We show that $S_j \subset l_a$. Indeed let $a_j - x = \sum \mu_i a_i$ be the representation of a point $a_j - x$ where $x \in S_j$. Let $\alpha = (a_i, a_j)$ where $i \neq j$. Then $(x, a_j) = 1 - \mu_j - \alpha \sum_{i \neq j} \mu_i \geq 1 - \sum \mu_i \geq 0$. Thus the construsted set A shows that $b_n \leq 1/(n+1)$.

Show now that $b_n \geq 1/(n+1)$. Consider an arbitrary set $A \in \mathcal{B}^n$. For every $a \in S$ and $0 \leq \varepsilon \leq 1/(n+1)$ put $A(a, \varepsilon) = \{x \in \mathbb{R}^n : \mu(A \cap (x + l_a)) \geq 1/(n+1) - \varepsilon\}$. Then every $A(a, \varepsilon)$ is a convex closet set and $\mu(A(a, \varepsilon) \cap A) = n/(n+1) + \varepsilon$. Now fix an arbitrary number $\varepsilon > 0$. Let $c_0, \dots, c_n \in S$. Since $\mu(A) > \sum \mu(A \setminus A(c_i, \varepsilon))$ then there exists a point $c(\varepsilon) \in A \cap \bigcap A(c_i, \varepsilon)$. Since A is a compact then there exists a cluster point c of the set $\{c(1/m) : m \in \mathbb{N}\}$. Then $c \in A \cap \bigcap A(c_i, 0)$. Then the Helly Theorem implies that $\bigcap \{A(c, 0) : c \in \mathbb{R}^n\} \neq \emptyset$. Therefore $b_n \geq 1/(n+1)$. \square

REFERENCES

- [G] B. Grünbaum, *Etudes on combinatorial geometry and theory of convex bodies*, M., Nauka, 1971. (in Russian)
- [S] H. Steinhaus, *Sto zadań*, M., Nauka, 1982. (in Russian)